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LOW FREQUENCY VIBRATIONS OF ELASTIC BARS

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Abstract—The method of expansion of three-dimensional displacements in a double power series of the transverse coordinates is employed to find one-dimensional equations applicable to low frequency vibrations of uniform, elastic, isotropic and anisotropic bars. The axial displacements accompanying torsion are chosen specially for each cross-sectional shape of bar—resulting in the correct, or nearly correct, torsional rigidity. Applications are to bars of elliptic, triangular and rectangular sections, illustrating various independent and coupled extensional, flexural and torsional modes of motion.

1. INTRODUCTION

Poisson's[1] reduction of the three-dimensional equations of linear elasticity to one-dimensional equations of low frequency extensional, flexural and torsional motions of bars, by means of series expansion followed by truncation, has been the subject of many revisions, modernizations, amplifications and extensions to higher frequencies—most recently, for example, by Volterra[2], Medick[3, 4], Bleustein and Stanley[5], Dökmeci[6] and A. E. Green, Naghdi and Wenner[7]. (References to earlier papers may be found in the review articles by W. A. Green[8], Abramson, Plass and Ripperger[9], Miklowitz[10] and Redwood[11]). Before proceeding with developments at high frequencies, there remains a modification that can be made at the low frequency end of the torsional part of the motion and also affects high frequencies—as will be shown subsequently.

In the linear theory, the usual process of truncation of a power series expansion of the displacements, u_i, $i = 1, 2, 3$, involves the retention of all or part of a sequence of early terms; e.g. if the axis of the bar is along x_1,

$$u_i = u_i^{0,0} + x_2 u_i^{1,0} + x_3 u_i^{0,1} + x_2 x_3 u_i^{1,1}, \tag{1.1}$$

where the $u_i^{m,n}$ are functions of x_1 and time, t, only. These terms readily accommodate low frequency extensional and flexural motions since, in those cases, the three-dimensional displacements approach the approximate forms, as the frequency approaches zero and the wave length approaches infinity, so that, at the limit, the extensional and flexural rigidities are exact. However, in the case of torsion, only the bar of elliptic (including circular) section has displacements that can be represented by (1.1), or part of it, at the zero frequency, long wave limit. For all other sections, the axial displacement is more complicated.

Since each section has its peculiar axial displacement in torsion, it is necessary to eschew a restriction to one or a few terms as a universal approximation for all sections if an *ad hoc* correction factor is to be avoided. The series of transverse components of displacement, $u_2^{m,n}$, $u_3^{m,n}$, may be truncated as usual, but an infinite series of axial components can be retained in the derivation of a set of approximate equations applicable to all sections. Then, for each section to which the equations are applied, a select few terms, not necessarily early ones, may be chosen judiciously. As has been shown[12], the selection can be made so that the exact, St. Venant torsional rigidity is obtained for the infinity of sections for which the St. Venant torsion function is a polynomial in the transverse coordinates[13]. For other sections, a close approximation to the St. Venant result may be found; for example, the two term selection $u_1^{1,3}$, $u_1^{3,1}$ produces an error of only 0·1% for the square section.

As usual, the order of the system of one-dimensional differential equations governing the motion is twice the number of components of displacement retained. However, the additional axial components do not contribute to the order of the final system as they are accompanied by an equal number of linear, algebraic equations relating them to the lower order terms. The solution of this system of simultaneous equations serves to eliminate the additional displacements—in effect allows their free development—and leaves, finally, only a single, second order differential equation governing the torsional motion. The scheme is analogous to the elimination of the contour strains in the usual low frequency approximation for extensional vibrations—which is also adopted here: the contour stresses (those in the normal plane) are set equal to zero and the resulting equations are used to eliminate the contour strains from the stress-strain relations; leading, ultimately, to a single, second order differential equation governing the axial displacement.

In addition to the second order differential equations for low frequency torsion and extension, there are two pairs of second order equations for flexure—also with vanishing contour stresses in the low frequency approximation. However, the Bernoulli-Euler equations (the most elementary form) are not good approximations to nearly as high frequencies as are the elementary equations for torsion and extension. Accordingly, the Timoshenko shear deformation terms [14], which do not raise the order of the differential equations, are retained to the end that, in the case of anisotropic bars in which flexure may be coupled with torsion and extension, the various equations are valid up to comparable frequencies.

The rotatory inertia terms which, together with the Timoshenko shear deformation, produce the fundamental, high frequency axial shear modes, are, in general, omitted from the equations of flexure as they introduce complications that are better left to a treatment of high frequency vibrations. In the first place, anisotropy may produce coupling between flexure and extension or torsion and, except possibly for torsion of circular or near circular sections, the approximations for extension and torsion are not valid to frequencies as high as those of the fundamental axial shear modes. Secondly, the shape of the cross-section of the bar may be such that a contour mode (one in which the displacement is in the plane of the normal section) may have a low frequency cut-off lower than that of either or both of the fundamental axial shear modes. An extreme example is that of a strip or blade for which one or even more than one of the contour flexure modes may have a lower frequency than that of even the lower of the fundamental axial shear modes. Since the contour stresses are neglected, in the low frequency approximation, there can be no contour modes and their absence is one of the factors which must be taken into account in deciding whether or not to include one or both of the rotatory inertia terms. Except for certain types of approximation valid only at very long wave lengths [15–17], it is not permissible to omit intermediate branches; and, of course, it is foolish to employ approximate equations which take into account branches higher than those of influence in the frequency range of interest. Medick, in his studies of high frequency extensional vibrations of isotropic bars of rectangular section [3, 4], appears to have been the first properly to take into account such complications.

In Section 2, which follows, there is summarized a now familiar process of obtaining a doubly infinite series of one-dimensional equations of motion, end conditions and stress-strain-displacement relations from the three dimensional equations of linear elasticity. Next, in Section 3, the truncation of the series for the low frequency approximation is effected. The main features of the truncation are: the elimination of the contour modes; the retention of the fundamental axial shear deformations but not, in general, the accompanying rotatory inertias; the retention of an unlimited number of axial displacement terms and the establishment of an equal number of linear algebraic equations to relate them to the lower order displacements. In Section 4, the six stress-equations of motion and boundary conditions are stated for the six surviving dependent variables: three components of displacement and three of rotation. The next three sections contain applications to extensional, torsional and flexural motions of isotropic bars of elliptic, equilateral triangular and rectangular cross-sections as examples, increasing in complexity, of the selection of terms from the infinite series of axial displacements. The calculations of the torsional rigidities have been described before [12], but are summarized here for completeness and ready reference. Section 8 contains a treatment of the bar of elliptic section and the most general crystal anisotropy—resulting in the coupling of all four modes of motion: extension, torsion and two flexures. This is followed, in Section 9, by a detailed application to a free-ended quartz bar, of elliptic section, with the trigonal axis and a digonal axis of elastic symmetry parallel to the principal axes of the ellipse—in which case one of the flexures is coupled with extension and the other with torsion, but the two pairs are not coupled. In Section 10 an analysis by Voigt [18, p. 641] is reviewed to explain why, except for the elliptic section, the absence of the contour modes restricts the allowable anisotropies to those for which the normal section of the bar is a plane of elastic symmetry. Such a bar, of rectangular section, is treated in the final section.

2. EXPANSION IN POWER SERIES

From the linear theory of elasticity we need, for the present purposes, only the stress-strain-displacement relations and a variational equation of motion. The former are

$$T_{ij} = c_{ijkl}S_{kl} \quad \text{or} \quad T_r = c_{rs}S_s, \tag{2.1}$$

$$S_{ij} = \tfrac{1}{2}(u_{j,i} + u_{i,j}), \tag{2.2}$$

in which the T_{ij}, $i,j = 1,2,3$ (or T_r, $r = 1\ldots 6$) are the components of stress; the S_{ij} (or S_s, $s = 1\ldots 6$) are the components of strain, where

$$(S_1, S_2, S_3) = (S_{11}, S_{22}, S_{33}), \quad (S_4, S_5, S_6) = 2(S_{23}, S_{31}, S_{12});$$

the u_i are the components of displacement; and the c_{ijkl} (or c_{rs}) are the elastic stiffnesses.

The variational equation of motion, for a body occupying a volume V, bounded by a surface S with outward normal $\mathbf{n}$ on which act surface tractions t_j, is[19, p. 115]

$$\int_{t_0}^{t_1} \mathrm{d}t \int_V (T_{ij,i} - \rho\ddot{u}_j)\delta u_j \,\mathrm{d}V + \int_{t_0}^{t_1} \mathrm{d}t \int_S (t_j - n_iT_{ij})\delta u_j \,\mathrm{d}S = 0, \tag{2.3}$$

from which follow the stress-equations of motion,

$$T_{ij,i} = \rho\ddot{u}_j, \tag{2.4}$$

and boundary conditions sufficient, in the absence of singularities and discontinuities, to assure a unique solution of the fifteen equations (2.1), (2.2) and (2.4): specification, at each point P, on S, of one member of each of the three products

$$T_{nn}u_n, \quad T_{ns}u_s, \quad T_{nt}u_t, \tag{2.5}$$

where n, s, t are orthogonal directions at P.

The cross-sections of the uniform bar are positioned with their centroids on the axis of x_1 and their principal axes along x_2 and x_3. The components of displacement are expanded in a double series of powers of the transverse coordinates:

$$u_i = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x_2{}^m x_3{}^n u_i{}^{m,n}(x_1, t) \equiv \sum_{m,n=0}^{\infty} x_2{}^m x_3{}^n u_i{}^{m,n}, \tag{2.6}$$

where $x_2{}^m$ and $x_3{}^n$ are the mth and nth powers of x_2 and x_3 while $u_i{}^{m,n}(x_1, t)$ is the amplitude (x_1 and t dependent) of the mnth term of the series. The single summation sign, for the double summation, is employed, in the sequel, to save space.

From (2.2) and (2.6), the components of strain may be expressed as

$$S_{ij} = \sum_{m,n=0}^{\infty} x_2{}^m x_3{}^n S_{ij}^{m,n} \quad \text{or} \quad S_r = \sum_{m,n=0}^{\infty} x_2{}^m x_3{}^n S_r{}^{m,n}, \tag{2.7}$$

where

$$S_{ij}^{m,n} = \tfrac{1}{2}[u_{j,i}^{m,n} + u_{i,j}^{m,n} + (m+1)(\delta_{2i}u_j{}^{m+1,n} + \delta_{2j}u_i{}^{m+1,n}) + (n+1)(\delta_{3i}u_j{}^{m,n+1} + \delta_{3j}u_i{}^{m,n+1})], \tag{2.8}$$

in which δ_{ij} is the Kronecker delta and the relation of the $S_r{}^{m,n}$ to the $S_{ij}^{m,n}$ is the same as that between the S_r and the S_{ij}. Note that, since the $u_i{}^{m,n}$ are independent of x_2 and x_3, the differential quotients $u_{j,i}^{m,n}$ are zero except if $i = 1$.

Upon substituting the expansion (2.6) in the variational equation (2.3), we find, for a bar with ends at $x_1 = \pm l$ and cross-sections with area A and contour C,

$$\sum_{m,n=0}^{\infty}\int_{t_0}^{t_1}\mathrm{d}t\int_{-l}^{l}(T_{1j}^{m,n}-mT_{2j}^{m-1,n}-nT_{3j}^{m,n-1}+F_j^{m,n}-\rho\ddot{U}^{m,n})\delta u_j^{m,n}\,\mathrm{d}x_1$$
$$+\sum_{m,n=0}^{\infty}\int_{t_0}^{t_1}\mathrm{d}t[t_j^{m,n}-T_{1j}^{m,n}]_{-l}^{l}\delta u_j^{m,n}=0, \tag{2.9}$$

where

$$T_{ij}^{m,n}=\int_A T_{ij}x_2^{\ m}x_3^{\ n}\,\mathrm{d}A, \tag{2.10}$$

$$F_j^{m,n}=\oint_C t_jx_2^{\ m}x_3^{\ n}\,\mathrm{d}s, \tag{2.11}$$

$$t_j^{m,n}=\int_A t_jx_2^{\ m}x_3^{\ n}\,\mathrm{d}A, \tag{2.12}$$

$$\ddot{U}_j^{m,n}=\sum_{p,q=0}^{\infty}I^{m+p,n+q}\ddot{u}_j^{p,q}, \tag{2.13}$$

$$I^{m+p,n+q}=\int_A x_2^{\ m+p}x_3^{\ n+q}\,\mathrm{d}A. \tag{2.14}$$

Note that, since x_2 and x_3 are principal axes of the section of the bar,

$$I^{1,0}=I^{0,1}=I^{1,1}=0. \tag{2.15}$$

In the case of a rectangular section $x_2=\pm b$, $x_3=\pm c$,

$$F_j^{m,n}=\int_{-c}^{c}[t_jx_2^{\ m}x_3^{\ n}]_{-b}^{b}\,\mathrm{d}x_3+\int_{-b}^{b}[t_jx_2^{\ m}x_3^{\ n}]_{-c}^{c}\,\mathrm{d}x_2 \tag{2.16}$$

and, for $m>1$ or $n>1$,

$$I^{m,n}=\begin{cases}4b^{m+1}c^{n+1}/(m+1)(n+1), & m \text{ and } n \text{ even}\\ 0, & m \text{ or } n \text{ odd.}\end{cases} \tag{2.17}$$

From (2.9) follow the stress-equations of motion of order m, n:

$$T_{1j,1}^{m,n}-mT_{2j}^{m-1,n}-nT_{3j}^{m,n-1}+F_j^{m,n}=\rho\ddot{U}^{m,n} \tag{2.18}$$

and the end conditions: one member of each of the products

$$T_{11}^{m,n}u_1^{\ m,n},\quad T_{12}^{m,n}u_2^{\ m,n},\quad T_{13}^{m,n}u_3^{\ m,n} \tag{2.19}$$

on $x_1=\pm l$.

Finally, the constitutive equations,

$$T_r^{m,n}=\sum_{p,q=0}^{\infty}I^{m+p,n+q}c_{rs}S_s^{\ p,q}, \tag{2.20}$$

are obtained by substituting (2.7) in (2.1) and the result in (2.10).

3. TRUNCATION OF SERIES FOR LOW FREQUENCIES

We begin the truncation by discarding, in the stress-strain relation (2.20), all $S_r^{\ m,n}$ for $m+n>1$ except those parts of $S_r^{\ m,n}$ which contain $u_1^{\ m,n}$ (but not $u_{1,1}^{m,n}$); i.e. we retain, at least temporarily, all orders of axial displacement and such displacements and displacement gradients as appear in the zero-order and first order strains: $S_r^{0,0}$, $S_r^{1,0}$, $S_r^{0,1}$. Thus, (2.20) becomes

$$T_r^{m,n}=\sum_{p,q=0}^{p+q<2}I^{m+p,n+q}c_{rs}S_s^{\ p,q}+\sum_{p+q>1}^{\infty}I^{m+p,n+q}[c_{r6}(p+1)u_1^{\ p+1,q}+c_{r5}(q+1)u_1^{\ p,q+1}]. \tag{3.1}$$

Next, in the expression (3.1) for the case $m = 0$, $n = 0$, i.e.

$$T_r^{0,0} = \sum_{p,q=0}^{p+q<2} I^{p,q} c_{rs} S_s^{p,q} + \sum_{p+q>1}^{\infty} I^{p,q} [c_{r6}(p+1) u_1^{p+1,q} + c_{r5}(q+1) u_1^{p,q+1}], \tag{3.2}$$

we set the zero-order contour stresses equal to zero:

$$T_2^{0,0} = T_3^{0,0} = T_4^{0,0} = 0 \tag{3.3}$$

and use the resulting equations to eliminate $S_2^{0,0}$, $S_3^{0,0}$ and $S_4^{0,0}$ from the zero-order stress-strain relations. By this process, we permit the free development of the zero-order contour strains. To obtain the solution of (3.3) for $T_1^{0,0}$, $T_5^{0,0}$ and $T_6^{0,0}$, we first multiply (3.2) by the compliances s_{rs}, where the matrix of the s_{rs} is the reciprocal of the stiffness matrix:

$$s_{rt} c_{rs} = \delta_{ts}, \quad \mathrm{r, s, t} = 1 \ldots 6, \tag{3.4}$$

to obtain

$$s_{rt} T_r^{0,0} = I^{0,0} S_t^{0,0} + \sum_{p+q>1}^{\infty} I^{p,q} [\delta_{t6}(p+1) u_1^{p+1,q} + \delta_{t5}(q+1) u_1^{p,q+1}]. \tag{3.5}$$

With (3.3), (3.5) become, for $t = 1, 5, 6$,

$$\begin{aligned} s_{11} T_1^{0,0} + s_{51} T_5^{0,0} + s_{61} T_6^{0,0} &= I^{0,0} S_1^{0,0}, \\ s_{15} T_1^{0,0} + s_{55} T_5^{0,0} + s_{65} T_6^{0,0} &= I^{0,0} S_5^{0,0} + \sum_{p+q>1}^{\infty} I^{p,q} (q+1) u_1^{p,q+1}, \\ s_{16} T_1^{0,0} + s_{56} T_5^{0,0} + s_{66} T_6^{0,0} &= I^{0,0} S_6^{0,0} + \sum_{p+q>1}^{\infty} I^{p,q} (p+1) u_1^{p+1,q}. \end{aligned} \tag{3.6}$$

The solution of (3.6) for $T_1^{0,0}$, $T_5^{0,0}$ and $T_6^{0,0}$ is

$$T_\alpha^{0,0} = I^{0,0} c_{\alpha\beta}^{0,0} S_\beta^{0,0} + \sum_{p+q>1}^{\infty} I^{p,q} [c_{\alpha 5}^{0,0}(q+1) u_1^{p,q+1} + c_{\alpha 6}^{0,0}(p+1) u_1^{p+1,q}], \quad \alpha,\beta = 1, 5, 6, \tag{3.7}$$

where

$$c_{\alpha\beta}^{0,0} = A_{\alpha\beta} / |s_{\alpha\beta}| \tag{3.8}$$

in which $|s_{\alpha\beta}|$ is the determinant

$$|s_{\alpha\beta}| = \begin{vmatrix} s_{11} & s_{15} & s_{16} \\ s_{51} & s_{55} & s_{56} \\ s_{61} & s_{65} & s_{66} \end{vmatrix} \tag{3.9}$$

and $A_{\alpha\beta}$ is the cofactor of element $\alpha\beta$ in $|s_{\alpha\beta}|$:

$$\begin{aligned} &A_{11} = s_{55}s_{66} - s_{56}^2, && A_{15} = s_{56}s_{61} - s_{51}s_{66}, && A_{16} = s_{51}s_{65} - s_{55}s_{61}, \\ &A_{51} = A_{15}, && A_{55} = s_{11}s_{66} - s_{16}^2, && A_{56} = s_{15}s_{61} - s_{11}s_{65}, \\ &A_{61} = A_{16}, && A_{65} = A_{65}, && A_{66} = s_{11}s_{55} - s_{15}^2. \end{aligned} \tag{3.10}$$

Note that the final, zero-order stress-strain relations (3.7) contain the axial tensile stress and strain:

$$T_1^{0,0}, \quad S_1^{0,0} = u_{1,1}^{0,0}, \tag{3.11}$$

for extensional motions, and the transverse shear stresses and the accompanying shear strains:

$$T_5^{0,0}, \quad S_5^{0,0} = u_{3,1}^{0,0} + u_1^{0,1}, \tag{3.12}$$

$$T_6^{0,0}, \quad S_6^{0,0} = u_{2,1}^{0,0} + u_1^{1,0}, \tag{3.13}$$

which participate in flexural motions. The strains $S_5^{0,0}$ and $S_6^{0,0}$ are the ones that are set equal to zero in the Bernoulli–Euler theory but are retained in Timoshenko's theory.

For later use, we record, here, the result, from (3.5), for the zero-order contour-shear strain:

$$S_4^{0,0} = 2S_{23}^{0,0} = (s_{14}T_1^{0,0} + s_{54}T_5^{0,0} + s_{64}T_6^{0,0})/I^{0,0}. \tag{3.14}$$

Proceeding, now, to the first order stress-strain relations, we retain only the flexural stresses and strains:

$$\begin{aligned} &T_1^{1,0}, \quad S_1^{1,0} = u_{1,1}^{1,0}, \\ &T_1^{0,1}, \quad S_1^{0,1} = u_{1,1}^{0,1}, \end{aligned} \tag{3.15}$$

and the torsional stresses and strains:

$$\begin{aligned} &T_5^{1,0}, \quad S_5^{1,0} = u_{3,1}^{1,0} + u_1^{1,1}, \\ &T_6^{0,1}, \quad S_6^{0,1} = u_{2,1}^{0,1} + u_1^{1,1}; \end{aligned} \tag{3.16}$$

i.e. we set

$$\begin{aligned} T_2^{1,0} = T_3^{1,0} = T_4^{1,0} = T_6^{1,0} = 0, \\ T_2^{0,1} = T_3^{0,1} = T_4^{0,1} = T_5^{0,1} = 0, \end{aligned} \tag{3.17}$$

in (3.1), and use these eight relations to eliminate (i.e. allow the free development of) the corresponding strains. Thus, we have, from (3.1), recalling that $I^{1,0} = I^{0,1} = I^{1,1} = 0$,

$$\begin{aligned} T_r^{1,0} &= I^{2,0}c_{rs}S_s^{1,0} + \sum_{p+q>1}^{\infty} I^{p+1,q}[c_{r6}(p+1)u_1^{p+1,q} + c_{r5}(q+1)u_1^{p,q+1}], \\ T_r^{0,1} &= I^{0,2}c_{rs}S_s^{0,1} + \sum_{p+q>1}^{\infty} I^{p,q+1}[c_{r6}(p+1)u_1^{p+1,q} + c_{r5}(q+1)u_1^{p,q+1}]; \end{aligned}$$

whence, upon multiplying by s_{rt}, we have

$$\begin{aligned} s_{rt}T_r^{1,0} &= I^{2,0}S_t^{1,0} + \sum_{p+q>1}^{\infty} I^{p+1,q}[\delta_{t6}(p+1)u_1^{p+1,q} + \delta_{t5}(q+1)u_1^{p,q+1}], \\ s_{rt}T_r^{0,1} &= I^{0,2}S_t^{0,1} + \sum_{p+q>1}^{\infty} I^{p,q+1}[\delta_{t6}(p+1)u_1^{p+1,q} + \delta_{t5}(q+1)u_1^{p,q+1}]. \end{aligned} \tag{3.18}$$

After expanding the left hand sides of (3.18) and employing (3.17), we can solve the first of (3.18) for $T_1^{1,0}$ and $T_5^{1,0}$ and the second of (3.18) for $T_1^{0,1}$ and $T_6^{0,1}$:

$$\begin{aligned} T_a^{1,0} &= I^{2,0}c_{ab}^{1,0}S_b^{1,0} + \sum_{p+q>1}^{\infty} I^{p+1,q}c_{a5}^{1,0}(q+1)u_1^{p,q+1}, \quad a, b = 1, 5, \\ T_c^{0,1} &= I^{0,2}c_{cd}^{0,1}S_d^{0,1} + \sum_{p+q>1}^{\infty} I^{p,q+1}c_{c6}^{0,1}(p+1)u_1^{p+1,q}, \; c, d = 1, 6, \end{aligned} \tag{3.19}$$

where $c_{ab}^{1,0} = c_{ba}^{1,0}$, $c_{cd}^{0,1} = c_{dc}^{0,1}$ and

$$(c_{11}^{1,0}, c_{55}^{1,0}, c_{15}^{1,0}) = (s_{55}, s_{11}, -s_{15})/(s_{11}s_{55} - s_{15}^2), \tag{3.20}$$

$$(c_{11}^{0,1}, c_{66}^{0,1}, c_{16}^{0,1}) = (s_{66}, s_{11}, -s_{16})/(s_{11}s_{66} - s_{16}^2). \tag{3.21}$$

Employing the strain-displacement relations in (3.11)–(3.13) and (3.15) and (3.16) and adopting the notations

$$u_5{}^{m,n} = u_3{}^{m,n}, \quad u_6{}^{m,n} = u_2{}^{m,n}, \tag{3.22}$$

we may write the zero-order and first-order stress-displacement relations, for the low frequency motions, as follows:

$$T_\alpha{}^{0,0} = I^{0,0} c_{\alpha\beta}^{0,0} u_{\beta,1}^{0,0} + \sum_{p,q=0}^{\infty} I^{p,q} [c_{\alpha 5}^{0,0}(q+1) u_1{}^{p,q+1} + c_{\alpha 6}^{0,0}(p+1) u_1{}^{p+1,q}], \quad \alpha,\beta = 1,5,6 \tag{3.23}$$

$$T_a{}^{1,0} = I^{2,0} c_{ab}^{1,0} u_{b,1}^{1,0} + \sum_{p,q=0}^{\infty} I^{p+1,q} c_{a5}^{1,0}(q+1) u_1{}^{p,q+1}, a, b = 1,5, \tag{3.24}$$

$$T_c{}^{0,1} = I^{0,2} c_{cd}^{0,1} u_{d,1}^{0,1} + \sum_{p,q=0}^{\infty} I^{p,q+1} c_{c6}^{0,1}(p+1) u_1{}^{p+1,q}, \quad c, d = 1,6. \tag{3.25}$$

As the final step in the truncation process, we allow the free development of the higher order axial warping of the sections of the bar, at the long wave, low frequency limit, by setting the force associated with each of the $u_1{}^{m,n}$, $m+n>1$, equal to zero. From the equations of motion (2.18), with $j = 1$, we omit the axial stress gradients $T_{11,1}^{m,n}$ and the accelerations $\ddot{U}_1{}^{m,n}$, which vanish at infinite wave length and zero frequency, and we also omit the surface forces $F_1{}^{m,n}$, with the result:

$$mT_6{}^{m-1,n} + nT_5{}^{m,n-1} = 0, m+n>1. \tag{3.26}$$

These are the equations which are to be used to eliminate the higher order $u_1{}^{m,n}$.

The stress-strain-displacement relations, for the stresses in (3.26), are obtained by setting

$$T_1{}^{m,n} = T_2{}^{m,n} = T_3{}^{m,n} = T_4{}^{m,n} = 0, m+n>1, \tag{3.27}$$

in (3.1) and solving for $T_5{}^{m,n}$ and $T_6{}^{m,n}$, with the result:

$$T_\xi{}^{m,n} = \sum_{p,q=0}^{p+q<2} I^{m+p,n+q} \bar{c}_{\xi\eta} S_\eta{}^{p,q} + \sum_{p+q>1}^{\infty} I^{m+p,n+q} [\bar{c}_{\xi 6}(p+1) u_1{}^{p+1,q} + \bar{c}_{\xi 5}(q+1) u_1{}^{p,q+1}], \tag{3.28}$$

where $\xi = 5, 6$; $\eta = 5, 6$ and

$$(\bar{c}_{55}, \bar{c}_{66}, \bar{c}_{56}) = (s_{66}, s_{55}, -s_{56})/(s_{55}s_{66} - s_{56}^2). \tag{3.29}$$

In (3.28), $S_5{}^{0,0}$, $S_6{}^{0,0}$, $S_5{}^{1,0}$ and $S_6{}^{0,1}$ are given in terms of displacements by (3.12), (3.13) and (3.16), but $S_6{}^{1,0}$ and $S_5{}^{0,1}$ have to be calculated from (3.18):

$$\begin{aligned} I^{2,0} S_6{}^{1,0} &= s_{a6} T_a{}^{1,0} - \sum_{p+q>1}^{\infty} I^{p+1,q}(p+1) u_1{}^{p+1,q}, a = 1,5, \\ I^{0,2} S_5{}^{0,1} &= s_{c5} T_c{}^{0,1} - \sum_{p+q>1}^{\infty} I^{p,q+1}(q+1) u_1{}^{p,q+1}, c = 1,6, \end{aligned} \tag{3.30}$$

where $T_a{}^{1,0}$ and $T_c{}^{0,1}$ are given by (3.19).

In the isotropic case,

$$s_{15} = s_{16} = s_{56} = 0, \quad s_{55} = s_{66} = 1/\mu, \quad s_{11} = 1/E, \tag{3.31}$$

where μ is the shear modulus and E is Young's modulus. Then (3.23)–(3.25) become

$$T_1^{0,0} = I^{0,0} E u_{1,1}^{0,0},$$

$$T_5^{0,0} = I^{0,0}\mu(u_{3,1}^{0,0} + u_1^{0,1}) + \mu \sum_{p+q>1}^{\infty} I^{p,q}(q+1)u_1^{p,q+1}, \tag{3.32}$$

$$T_6^{0,0} = I^{0,0}\mu(u_{2,1}^{0,0} + u_1^{1,0}) + \mu \sum_{p+q>1}^{\infty} I^{p,q}(p+1)u_1^{p+1,q},$$

$$T_1^{1,0} = I^{2,0} E u_{1,1}^{1,0}, \tag{3.33}$$

$$T_5^{1,0} = I^{2,0}\mu u_{3,1}^{1,0} + \mu \sum_{p,q=0}^{\infty} I^{p+1,q}(q+1)u_1^{p,q+1},$$

$$T_1^{0,1} = I^{0,2} E u_{1,1}^{0,1}, \tag{3.34}$$

$$T_6^{0,1} = I^{0,2}\mu u_{2,1}^{0,1} + \mu \sum_{p,q=0}^{\infty} I^{p,q+1}(p+1)u_1^{p+1,q},$$

and, from (3.14), $S_4^{0,0} = 0$. Also, (3.28) and (3.30) become

$$T_5^{m,n} = \mu \sum_{p,q=0}^{p+q<2} I^{m+p,n+q} S_5^{p,q} + \mu \sum_{p+q>1}^{\infty} I^{m+p,n+q}(q+1)u_1^{p,q+1},\ m+n>1, \tag{3.35}$$

$$T_6^{m,n} = \mu \sum_{p,q=0}^{p+q<2} I^{m+p,n+q} S_6^{p,q} + \mu \sum_{p+q>1}^{\infty} I^{m+p,n+q}(p+1)u_1^{p+1,q},\ m+n>1$$

and

$$I^{2,0} S_6^{1,0} = -\sum_{p+q>1}^{\infty} I^{p+1,q}(p+1)u_1^{p+1,q} \tag{3.36}$$

$$I^{0,2} S_5^{0,1} = -\sum_{p+q>1}^{\infty} I^{p,q+1}(q+1)u_1^{p,q+1}$$

respectively.

4. STRESS-EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

There is a stress-equation of motion (or equilibrium) for each of the surviving components of displacement and rotation.

For the zero-order displacements $u_1^{0,0}$, $u_2^{0,0}$, $u_3^{0,0}$, we have, from (2.18) with $m = 0$, $n = 0$, the stress-equations of motion

$$T_{1,1}^{0,0} + F_1^{0,0} = \rho I^{0,0} \ddot{u}_1^{0,0},$$

$$T_{6,1}^{0,0} + F_2^{0,0} = \rho I^{0,0} \ddot{u}_2^{0,0}, \tag{4.1}$$

$$T_{5,1}^{0,0} + F_3^{0,0} = \rho I^{0,0} \ddot{u}_3^{0,0},$$

in which the higher order accelerations have been omitted.

For the rotations $u_1^{1,0}$ and $u_1^{0,1}$ about x_3 and x_2, respectively, we have, from (2.18) with $m = 1$, $n = 0$ and $m = 0$, $n = 1$ and $j = 1$, the stress-equations of equilibrium

$$T_{1,1}^{1,0} - T_6^{0,0} + F_1^{1,0} = 0,$$

$$T_{1,1}^{0,1} - T_5^{0,0} + F_1^{0,1} = 0, \tag{4.2}$$

in which the rotatory inertias $\rho I^{2,0}\ddot{u}_1^{1,0}$ and $\rho I^{0,2}\ddot{u}_1^{0,1}$ and all the higher order inertias have been omitted.

The rotation about x_1 (the torsional rotation) is defined as

$$\theta = \tfrac{1}{2}(u_3{}^{1,0} - u_2{}^{0,1}) \tag{4.3}$$

(i.e. positive clockwise advancing in the positive direction along x_1) and we note that, since $S_{23}^{0,0} = \frac{1}{2}(u_3{}^{1,0} + u_2{}^{0,1})$,

$$u_3{}^{1,0} = S_{23}^{0,0} + \theta, \quad u_2{}^{0,1} = S_{23}^{0,0} - \theta. \tag{4.4}$$

The torsional stress-equation of motion is obtained by subtracting (2.18), with $m = 0$, $n = 1$, $j = 2$, from (2.18), with $m = 1$, $n = 0$, $j = 3$, and neglecting the contour shear acceleration $\ddot{S}_{2,3}^{0,0}$ and all the higher order accelerations. The result is

$$T_{5,1}^{1,0} - T_{6,1}^{0,1} + F_\theta = \rho(I^{2,0} + I^{0,2})\ddot{\theta}, \tag{4.5}$$

in which $F_\theta = F_3{}^{1,0} - F_2{}^{0,1}$.

The torque about the line of centroids of the bar is

$$M_1 = T_5{}^{1,0} - T_6{}^{0,1}, \tag{4.6}$$

positive M_1 producing positive θ. The couples $T_1{}^{1,0}$ and $T_1{}^{0,1}$ are the bending moments sometimes designated by

$$M_3 = -T_1{}^{1,0}, \quad M_2 = T_1{}^{0,1}, \tag{4.7}$$

i.e. positive counter-clockwise about the axes of x_3 and x_2, respectively, as viewed by an observer looking toward the origin.

The equilibrium equations (3.26), one for each $u_1{}^{m,n}$, $m + n > 1$, that is retained, constitute a set of linear algebraic equations which are to be solved for the $u_1{}^{m,n}$ in terms of the six dependent variables: the three displacements $u_1{}^{0,0}$, $u_2{}^{0,0}$, $u_3{}^{0,0}$ and the three rotations $u_1{}^{1,0}$, $u_1{}^{0,1}$, θ. The resulting expressions are then to be substituted in the constitutive equations (3.28), for the anisotropic case, or (3.32)–(3.34) for the isotropic case, to obtain the stresses exclusively in terms of the six dependent variables. This will permit the six stress-equations of motion, (4.1), (4.2) and (4.5), to be expressed solely in terms of the six variables—resulting in a twelfth order system of partial differential equations for the general, anisotropic case.

As for end conditions, all that remains of (2.19) is the prescription, at $x_1 = \pm l$, of one member of each of the six products

$$T_1{}^{0,0}u_1{}^{0,0}, \quad T_6{}^{0,0}u_2{}^{0,0}, \quad T_5{}^{0,0}u_3{}^{0,0}, \quad T_1{}^{1,0}u_1{}^{1,0}, \quad T_1{}^{0,1}u_1{}^{0,1}, \quad (T_5{}^{1,0} - T_6{}^{0,1})\theta. \tag{4.8}$$

There are also six surface-tractions:

$$F_1{}^{0,0}, \quad F_2{}^{0,0}, \quad F_3{}^{0,0}, \quad F_1{}^{1,0}, \quad F_1{}^{0,1}, \quad F_\theta, \tag{4.9}$$

to be prescribed on the cylindrical or prismatic surface of the bar. Alternatively, one or more of the six dependent variables may be specified—as a consequence of which the order of the system of differential equations is reduced, as each of the corresponding equations becomes simply a formula for a surface traction.

5. ISOTROPIC BAR, ELLIPTIC SECTION

In this and the next two articles, the equations of motion are derived for isotropic bars of elliptic, triangular and rectangular sections to illustrate successively more complicated contributions of the terms of the series of axial displacements.

For the ellipse, with semi-principal axes b and c, along x_2 and x_3, respectively, the axial displacement found from the St. Venant theory is [19, p. 317]

$$u_1 = -(b^2 - c^2)\theta_{,1}x_2x_3/(b^2 + c^2). \tag{5.1}$$

Accordingly, from the series of $u_1^{m,n}$, $m+n>1$, we retain only $u_1^{1,1}$. Then (3.26) become

$$T_6^{0,1}+T_5^{1,0}=0 \tag{5.2}$$

and (3.32)–(3.34) reduce to

$$\begin{aligned}
T_1^{0,0}&=I^{0,0}Eu_{1,1}^{0,0}\\
T_6^{0,0}&=I^{0,0}\mu(u_{2,1}^{0,0}+u_1^{1,0}),\\
T_5^{0,0}&=I^{0,0}\mu(u_{3,1}^{0,0}+u_1^{0,1}),\\
T_1^{1,0}&=I^{2,0}Eu_{1,1}^{1,0},\\
T_5^{1,0}&=I^{2,0}\mu(u_{3,1}^{1,0}+u_1^{1,1})=I^{2,0}\mu(u_1^{1,1}+\theta_{,1}),\\
T_1^{0,1}&=I^{0,2}Eu_{1,1}^{0,1},\\
T_6^{0,1}&=I^{0,2}\mu(u_{2,1}^{0,1}+u_1^{1,1})=I^{0,2}\mu(u_1^{1,1}-\theta_{,1}).
\end{aligned} \tag{5.3}$$

The second forms for $T_5^{1,0}$ and $T_6^{1,0}$ are obtained from (4.4), in which $S_{23}^{0,0}$, according to (3.14), is zero for the isotropic case. Upon substituting these two expressions in (5.2) and solving for $u_1^{1,1}$, we find

$$u_1^{1,1}=(I^{0,2}-I^{2,0})\theta_{,1}/(I^{0,2}+I^{2,0}), \tag{5.4}$$

from which the torque-twist relation is

$$T_5^{1,0}-T_6^{0,1}=4\mu I^{0,2}I^{2,0}\theta_{,1}/(I^{0,2}+I^{2,0}). \tag{5.5}$$

From (2.14),

$$I^{0,2}=\pi bc^3/4,\quad I^{2,0}=\pi b^3c/4. \tag{5.6}$$

Hence, the torsional rigidity is

$$C=(T_5^{1,0}-T_6^{0,1})/\theta_{,1}=\pi\mu b^3c^3/(b^2+c^2), \tag{5.7}$$

which is the St. Venant value ([19], p. 323).

The equation of torsional motion is, from (5.5) and (4.5),

$$4\mu I^{0,2}I^{2,0}\theta_{,11}+(I^{0,2}+I^{2,0})F_\theta=\rho(I^{0,2}+I^{2,0})^2\ddot{\theta}. \tag{5.8}$$

The equations of extensional and flexural motion are, from (5.3), (4.1) and (4.2):

Extensional,

$$I^{0,0}Eu_{1,11}^{0,0}+F_1^{0,0}=\rho I^{0,0}\ddot{u}_1^{0,0}, \tag{5.9}$$

Flexural (x_1-x_2 plane),

$$\begin{aligned}
&I^{0,0}\mu(u_{2,11}^{0,0}+u_{1,1}^{1,0})+F_2^{0,0}=\rho I^{0,0}\ddot{u}_2^{0,0},\\
&I^{2,0}Eu_{1,11}^{1,0}-I^{0,0}\mu(u_{2,1}^{0,0}+u_1^{1,0})+F_1^{1,0}=0,
\end{aligned} \tag{5.10}$$

Flexural (x_1-x_3 plane),

$$\begin{aligned}
&I^{0,0}\mu(u_{3,11}^{0,0}+u_{1,1}^{0,1})+F_3^{0,0}=\rho I^{0,0}\ddot{u}_3^{0,0},\\
&I^{0,2}Eu_{1,11}^{0,1}-I^{0,0}\mu(u_{3,1}^{0,0}+u_1^{0,1})+F_1^{0,1}=0.
\end{aligned} \tag{5.11}$$

There is no coupling between extensional, flexural and torsional motions and the torsional warping of the sections does not affect the extensional and flexural stiffnesses. Equations (5.10) and (5.11) are Timoshenko's beam equations[14] with the rotatory inertia terms and the shear-correction factors omitted. If the eccentricity of the elliptic cross-section is not large enough for the lowest contour-flexure mode to approach one of the fundamental axial shear modes, the rotatory inertia terms $\rho I^{2,0}\ddot{u}_1{}^{1,0}$ and $\rho I^{0,2}\ddot{u}_1{}^{0,1}$ may be added to the right hand sides of the rotational equations in (5.10) and (5.11). Concurrently, the Timoshenko shear-correction factors should be introduced by replacing μ by $\kappa_2{}^2\mu$ in (5.10) and by $\kappa_3{}^2\mu$ in (5.11) where κ_2 and κ_3 are chosen to make the frequencies of the axial shear modes match the corresponding ones of the three-dimensional theory[20]. The ranges of validity of the equations of flexure are thereby raised to frequencies about 20% above those of the axial shear modes. With the rotatory inertia terms omitted, "20% above" is reduced to about "50% below" and a slight improvement can be effected by retaining the shear-correction factor but adjusting it to obtain a match between the surface wave frequencies from the approximate and exact equations[21].

The lower the frequency, the less is the influence of the shear-deformation terms and, hence, the shear-correction factor. At low enough frequencies, say less than 10% of the fundamental axial shear frequency, the shear-deformation terms may be omitted entirely: thereby reducing (5.10) and (5.11) to Bernoulli-Euler equations. This is done by eliminating the shear terms

$$u_{2,1}^{0,0} + u_1{}^{1,0}, \quad u_{3,1}^{0,0} + u_1{}^{0,1}$$

between the deflection and rotation equations, following which the rotations $u_1{}^{1,0}$ and $u_1{}^{0,1}$ are replaced by $-u_{2,1}^{0,0}$ and $-u_{3,1}^{0,0}$, respectively, with the results:

$$I^{2,0}Eu_{2,1111}^{0,0} + \rho I^{0,0}\ddot{u}_2{}^{0,0} = F_2{}^{0,0} + F_{1,1}^{1,0}, \tag{5.10$'$}$$

$$I^{0,2}Eu_{3,1111}^{0,0} + \rho I^{0,0}\ddot{u}_3{}^{0,0} = F_3{}^{0,0} + F_{1,1}^{0,1}, \tag{5.11$'$}$$

which are the Bernoulli–Euler equations.

6. ISOTROPIC BAR, TRIANGULAR SECTION

If the cross-section of the isotropic bar is an equilateral triangle with sides defined by[19, p. 319]

$$(x_2 - a)(x_2 - x_3\sqrt{3} + 2a)(x_2 + x_3\sqrt{3} + 2a) = 0, \tag{6.1}$$

the axial displacement in torsion, according to St. Venant, is[19, p. 319]

$$u_1 = (3x_2{}^2x_3 - x_3{}^3)\theta_{,1}/6a. \tag{6.2}$$

Therefore, of the $u_1{}^{m,n}$, $m + n > 1$, we retain only $u_1{}^{2,1}$ and $u_1{}^{0,3}$; and the equations for calculating them are, from (3.26) with $m = 2$, $n = 1$ and $m = 0$, $n = 3$:

$$2T_6{}^{1,1} + T_5{}^{2,0} = 0, \quad T_5{}^{0,2} = 0. \tag{6.3}$$

For the stresses, we have, from (3.32)–(3.34),

$$\begin{aligned}
T_1{}^{0,0} &= I^{0,0}Eu_{1,1}^{0,0},\\
T_5{}^{0,0} &= \mu[I^{0,0}(u_{3,1}^{0,0} + u_1{}^{0,1}) + I^{2,0}u_1{}^{2,1} + 3I^{0,2}u_1{}^{0,3}],\\
T_6{}^{0,0} &= \mu I^{0,0}(u_{2,1}^{0,0} + u_1{}^{1,0}),\\
T_1{}^{1,0} &= I^{2,0}Eu_{1,1}^{1,0},\\
T_5{}^{1,0} &= \mu(I^{2,0}u_{3,1}^{1,0} + I^{3,0}u_1{}^{2,1} + 3I^{1,2}u_1{}^{0,3}),\\
T_1{}^{0,1} &= I^{0,2}Eu_{1,1}^{0,1},\\
T_6{}^{0,1} &= \mu(I^{0,2}u_{2,1}^{0,1} + 2I^{1,2}u_1{}^{2,1}),
\end{aligned} \tag{6.4}$$

and, from (3.35), noting that, in the present case, $I^{m,n}=0$ if n is odd,

$$\begin{aligned} T_5^{2,0} &= \mu[I^{2,0}(u_{3,1}^{0,0}+u_1^{0,1})+I^{3,0}u_{3,1}^{1,0}+I^{4,0}u_1^{2,1}+3I^{2,2}u_1^{0,3}], \\ T_5^{0,2} &= \mu[I^{0,2}(u_{3,1}^{0,0}+u_1^{0,1})+I^{1,2}u_{3,1}^{1,0}+I^{2,2}u_1^{2,1}+3I^{0,4}u_1^{0,3}], \\ T_6^{1,1} &= \mu(I^{1,2}u_{2,1}^{0,1}+2I^{2,2}u_1^{2,1}), \end{aligned} \tag{6.5}$$

in which, from (6.1) and (2.14),

$$\begin{aligned} I^{0,0} &= 3\sqrt{3}a^2, \\ I^{0,2} &= I^{2,0} = 3\sqrt{3}a^4/2, \\ I^{1,2} &= -I^{3,0} = 3\sqrt{3}a^5/5, \\ I^{0,4} &= I^{4,0} = 3I^{2,2} = 9\sqrt{3}a^6/5. \end{aligned} \tag{6.6}$$

Upon substituting (6.6) in (6.5) and the result in (6.3), we may solve the latter for $u_1^{2,1}$ and $u_1^{0,3}$:

$$\begin{aligned} u_1^{2,1} &= \theta_{,1}/2a-(u_{3,1}^{0,0}+u_1^{0,1})/4a^2, \\ u_1^{0,3} &= -\theta_{,1}/6a-(u_{3,1}^{0,0}+u_1^{0,1})/4a^2, \end{aligned} \tag{6.7}$$

after noting, from (4.4) and (3.14), that $u_3^{1,0}=-u_2^{0,1}=\theta$. Then, inserting (6.7) in the second, fifth and seventh of (6.4) we find

$$\begin{aligned} T_5^{0,0} &= \frac{1}{2}I^{0,0}\mu(u_{3,1}^{0,0}+u_1^{0,1}), \\ T_5^{1,0}-T_6^{0,1} &= I^{0,4}\mu\theta_{,1}/a^2. \end{aligned} \tag{6.8}$$

The second of (6.8), with (6.6), yields the torsional rigidity:

$$C=(T_5^{1,0}-T_6^{0,1})/\theta_{,1}=9\sqrt{3}\mu a^4/5, \tag{6.9}$$

which is the result from the St. Venant theory [22, p. 266]; and the equation of torsional motion is, from (6.8), (6.6) and (4.5),

$$\mu I^{0,4}\theta_{,11}+a^2F_\theta=2a^2I^{0,2}\rho\ddot{\theta}. \tag{6.10}$$

The equations of extensional and flexural motions are, from (6.4), (6.7), (4.1) and (4.2),

Extensional,

$$I^{0,0}Eu_{1,11}^{0,0}+F_1^{0,0}=\rho I^{0,0}\ddot{u}_1^{0,0}, \tag{6.11}$$

Flexural (x_1-x_2 *plane*),

$$\begin{aligned} I^{0,0}\mu(u_{2,11}^{0,0}+u_{1,1}^{1,0})+F_2^{0,0} &= \rho I^{0,0}\ddot{u}_2^{0,0}, \\ I^{2,0}Eu_{1,11}^{1,0}-I^{0,0}\mu(u_{2,1}^{0,0}+u_1^{1,0})+F_1^{1,0} &= 0, \end{aligned} \tag{6.12}$$

Flexural (x_1-x_3 *plane*),

$$\begin{aligned} \frac{1}{2}I^{0,0}\mu(u_{3,11}^{0,0}+u_{1,1}^{0,1})+F_{3,}^{0,0} &= \rho I^{0,0}\ddot{u}_3^{0,0}, \\ I^{0,2}Eu_{1,11}^{0,1}-\frac{1}{2}I^{0,0}\mu(u_{3,1}^{0,0}+u_1^{0,1})+F_1^{0,1} &= 0. \end{aligned} \tag{6.13}$$

Inclusion of the axial displacements $u_1^{2,1}$ and $u_1^{0,3}$ does not affect the flexural motion in the $x_1 - x_2$ plane—which is a plane of geometrical symmetry—but the flexural motion in the $x_1 - x_3$ plane (parallel to a face of the triangular prism) *is* affected, as can be seen by the factor 1/2 in the expression for $T_5^{0,0}$ in terms of displacements, in (6.8), and, consequently, in (6.13). This effect diminishes as the frequency approaches zero, so that the two dispersion relations become the same.

If the rotatory inertia terms are included, the two shear-correction factors for flexure may be calculated from the circular frequency

$$\omega = (2\pi/3a)(\mu/3\rho)^{\frac{1}{2}} \tag{6.14}$$

(the same for both modes) obtained from the three-dimensional theory[23]. As a result, the two dispersion relations again become the same.

7. ISOTROPIC BAR, RECTANGULAR SECTION

For the rectangular section, the axial displacement in St. Venant torsion is not expressible as a finite polynomial. As an approximation we retain, of the $u_1^{m,n}$, $m + n > 1$, only the four terms

$$u_1^{1,1}, \quad u_1^{1,3}, \quad u_1^{3,1}, \quad u_1^{3,3} \tag{7.1}$$

so that (3.26) become

$$\begin{aligned} T_6^{0,1} + T_5^{1,0} &= 0, \\ T_6^{0,3} + 3T_5^{1,2} &= 0, \\ 3T_6^{2,1} + T_5^{3,0} &= 0, \\ T_6^{2,3} + T_5^{3,2} &= 0; \end{aligned} \tag{7.2}$$

and the stresses, from (3.32)–(3.34) and (3.35) are

$$\begin{aligned} T_1^{0,0} &= I^{0,0}Eu_{1,1}^{0,0}, \\ T_5^{0,0} &= I^{0,0}\mu(u_{3,1}^{0,0} + u_1^{0,1}), \\ T_6^{0,0} &= I^{0,0}\mu(u_{2,1}^{0,0} + u_1^{1,0}), \\ T_1^{1,0} &= I^{2,0}Eu_{1,1}^{1,0}, \\ T_1^{0,1} &= I^{0,2}Eu_{1,1}^{0,1}, \end{aligned} \tag{7.3}$$

and

$$\begin{aligned} T_6^{0,1} &= \mu(-I^{0,2}\theta_{,1} + I^{0,2}u_1^{1,1} + I^{0,4}u_1^{1,3} + 3I^{2,2}u_1^{3,1} + 3I^{2,4}u_1^{3,3}), \\ T_5^{1,0} &= \mu(I^{2,0}\theta_{,1} + I^{2,0}u_1^{1,1} + 3I^{2,2}u_1^{1,3} + I^{4,0}u_1^{3,1} + 3I^{4,2}u_1^{3,3}), \\ T_6^{0,3} &= \mu(-I^{0,4}\theta_{,1} + I^{0,4}u_1^{1,1} + I^{0,6}u_1^{1,3} + 3I^{2,4}u_1^{3,1} + 3I^{2,6}u_1^{3,3}), \\ T_5^{1,2} &= \mu(I^{2,2}\theta_{,1} + I^{2,2}u_1^{1,1} + 3I^{2,4}u_1^{1,3} + I^{4,2}u_1^{3,1} + 3I^{4,4}u_1^{3,3}), \\ T_6^{2,1} &= \mu(-I^{2,2}\theta_{,1} + I^{2,2}u_1^{1,1} + I^{2,4}u_1^{1,3} + 3I^{4,2}u_1^{3,1} + 3I^{4,4}u_1^{3,3}), \\ T_5^{3,0} &= \mu(I^{4,0}\theta_{,1} + I^{4,0}u_1^{1,1} + 3I^{4,2}u_1^{1,3} + I^{6,0}u_1^{3,1} + 3I^{6,2}u_1^{3,3}), \\ T_6^{2,3} &= \mu(-I^{2,4}\theta_{,1} + I^{2,4}u_1^{1,1} + I^{2,6}u_1^{1,3} + 3I^{4,4}u_1^{3,1} + 3I^{4,6}u_1^{3,3}), \\ T_5^{3,2} &= \mu(I^{4,2}\theta_{,1} + I^{4,2}u_1^{1,1} + 3I^{4,4}u_1^{1,3} + I^{6,2}u_1^{3,1} + 3I^{6,4}u_1^{3,3}), \end{aligned} \tag{7.4}$$

in which the $I^{m,n}$ are calculated from (2.14) or (2.17) for the section $x_2 = \pm b$, $x_3 = \pm c$.

Upon substituting (7.4) in (7.2) and solving for the axial displacements, we find

$$u_1^{1,1} = (k^4-1)(14+375k^2+14k^4)\theta_{,1}/\Delta,$$
$$u_1^{1,3} = 35k^2(1-6k^2-7k^4)\theta_{,1}/c^2\Delta,$$
$$u_1^{3,1} = 35k^2(7+6k^2-k^4)\theta_{,1}/b^2\Delta, \quad (7.5)$$
$$u_1^{3,3} = 245k^2(k^4-1)\theta_{,1}/3b^2c^2\Delta,$$

where

$$k = c/b, \quad \Delta = 2(k^2+1)^2(7+121k^2+7k^4). \quad (7.6)$$

The torsional rigidity is

$$C = (T_5^{1,0} - T_6^{0,1})/\theta_{,1} = k_1\mu(2b)^3(2c), \quad (7.7)$$

where

$$12k_1 = k^2+1-\frac{(k^2-1)^2(7+212k^2+7k^4)}{(k^2+1)(7+121k^2+7k^4)}+\frac{7k^2(13-25k^2-25k^4+13k^6)}{(k^2+1)^2(7+121k^2+7k^4)}. \quad (7.8)$$

Values of k_1 for various values of k are listed in Table 1 along with the corresponding

Table 1. Torsional rigidity coefficient for rectangular section: k_1: four terms of power series, k_1': St. Venant solution

$k = c/b$	k_1	k_1'	$k = c/b$	k_1	k_1'
1.0	0.1407	0.1406	4	0.286	0.281
1.2	0.166	0.166	5	0.298	0.291
1.5	0.196	0.196	10	0.322	0.312
2.0	0.230	0.229	20	0.330	0.323
2.5	0.251	0.249	100	0.333	0.331
3.0	0.266	0.263	∞	1/3	1/3

values[22, p. 277] calculated from St. Venant's solution. It may be seen, from (7.5), that, for the square section ($k = 1$), $u_1^{1,1}$ and $u_1^{3,3}$ are zero and the approximation reduces to

$$u_1^{3,1} = -u_1^{1,3} = 7/18b^2, \quad k_1 = 19/135. \quad (7.9)$$

This simple approximation yields a torsional rigidity differing from St. Venant's result by only about 0·1%. The fact that $u_1^{1,1}$ is zero for the square section accounts for the failure of the Bleustein-Stanley[5] equations for that section, as they used only the term $u_1^{1,1}$ in their approximation for the axial displacement. As k increases from unity, the error in torsional rigidity increases to about 3% around $k = 10$ and then diminishes to zero as k approaches infinity; at which limit $u_1^{1,3}$, $u_1^{3,1}$ and $u_1^{3,3}$, in (7.5), are zero and $u_1^{1,1} = \theta_{,1}$, corresponding to the correct form: $u_1 = x_2x_3\theta_{,1}$ of the St. Venant solution at that limit.

The equation of torsional motion is, from (7.7) and (4.5),

$$16k_1\mu b^3c\theta_{,11} + F_\theta = 2bc(b^2+c^2)\rho\ddot{\theta}. \quad (7.10)$$

The equations of extensional and flexural motion are the same as (5.9), (5.10) and (5.11) for the elliptic section—except that $I^{0,0}$, $I^{0,2}$ and $I^{2,0}$ are given by (2.14).

8. ANISOTROPIC BAR, ELLIPTIC SECTION

The bar treated in this article has the same section as in Section 5, but the material is now anisotropic. According to the St. Venant torsion theory, the axial displacement is

$$u_1 = \left(\frac{s_{15}M_1}{2I^{2,0}} - \frac{s_{11}M_3}{I^{2,0}}\right)x_1x_2 - \left(\frac{s_{16}M_1}{2I^{0,2}} - \frac{s_{11}M_2}{I^{0,2}}\right)x_1x_3 + \left(\frac{s_{56}M_1}{4I^{2,0}} + \frac{s_{16}M_3}{2I^{2,0}}\right)x_2^2$$
$$-\left(\frac{s_{56}M_1}{4I^{0,2}} - \frac{s_{15}M_2}{2I^{0,2}}\right)x_3^2 + \left[\frac{M_1}{4}\left(\frac{s_{55}}{I^{2,0}} - \frac{s_{66}}{I^{0,2}}\right) + \frac{s_{16}M_2}{2I^{0,2}} - \frac{s_{15}M_3}{2I^{2,0}}\right]x_2x_3 \quad (8.1)$$

and the twist is[18, p. 638]

$$\theta_{,1} = \left(\frac{s_{55}}{I^{2,0}} + \frac{s_{66}}{I^{0,2}}\right)\frac{M_1}{4} - \frac{s_{16}M_2}{2I^{0,2}} - \frac{s_{15}M_3}{2I^{2,0}}, \tag{8.2}$$

where M_1 is the torque and M_2 and M_3 are the bending moments as defined in (4.6) and (4.7). From (8.1), we conclude that we have to retain $u_1^{1,1}$, $u_1^{0,2}$ and $u_1^{2,0}$ in the power series for $u_1^{m,n}$, $m+n>1$, and, hence, the equilibrium equations

$$T_6^{0,1} + T_5^{1,0} = 0, \quad T_6^{1,0} = 0, \quad T_5^{0,1} = 0 \tag{8.3}$$

from (3.26). Now, as may be seen in (3.17) *et seq.*, the last two of (8.3), which provide for the free development of $u_1^{0,2}$ and $u_1^{2,0}$, have already been incorporated in the truncated equations. Hence, we need apply only the first of (8.3) and include only $u_1^{1,1}$ in the series of $u_1^{m,n}$, $m+n>1$, in the expressions (3.23), (3.24) and (3.25), for the stresses, which become:

$$\begin{aligned} T_1^{0,0} &= I^{0,0}(c_{11}^{0,0}u_{1,1}^{0,0} + c_{15}^{0,0}u_{3,1}^{0,0} + c_{16}^{0,0}u_{2,1}^{0,0} + c_{15}^{0,0}u_1^{0,1} + c_{16}^{0,0}u_1^{1,0}),\\ T_5^{0,0} &= I^{0,0}(c_{51}^{0,0}u_{1,1}^{0,0} + c_{55}^{0,0}u_{3,1}^{0,0} + c_{56}^{0,0}u_{2,1}^{0,0} + c_{55}^{0,0}u_1^{0,1} + c_{56}^{0,0}u_1^{1,0}),\\ T_6^{0,0} &= I^{0,0}(c_{61}^{0,0}u_{1,1}^{0,0} + c_{65}^{0,0}u_{3,1}^{0,0} + c_{66}^{0,0}u_{2,1}^{0,0} + c_{65}^{0,0}u_1^{0,0} + c_{66}^{0,0}u_1^{1,0}), \end{aligned} \tag{8.4}$$

$$\begin{aligned} T_1^{1,0} &= I^{2,0}[c_{11}^{1,0}u_{1,1}^{1,0} + c_{15}^{1,0}(u_{3,1}^{1,0} + u_1^{1,1})] = I^{2,0}[c_{11}^{1,0}u_{1,1}^{1,0} + c_{15}^{1,0}(\hat{u}_1^{1,1} + \theta_{,1})],\\ T_5^{1,0} &= I^{2,0}[c_{51}^{1,0}u_{1,1}^{1,0} + c_{55}^{1,0}(u_{3,1}^{1,0} + u_1^{1,1})] = I^{2,0}[c_{51}^{1,0}u_{1,1}^{1,0} + c_{55}^{1,0}(\hat{u}_1^{1,1} + \theta_{,1})], \end{aligned} \tag{8.5}$$

$$\begin{aligned} T_1^{0,1} &= I^{0,2}[c_{11}^{0,1}u_{1,1}^{0,1} + c_{16}^{0,1}(u_{2,1}^{0,1} + u_1^{1,1})] = I^{0,2}[c_{11}^{0,1}u_{1,1}^{0,1} + c_{16}^{0,1}(\hat{u}_1^{1,1} - \theta_{,1})],\\ T_6^{0,1} &= I^{0,2}[c_{61}^{0,1}u_{1,1}^{0,1} + c_{66}^{0,1}(u_{2,1}^{0,1} + u_1^{1,1})] = I^{0,2}[c_{61}^{0,1}u_{1,1}^{0,1} + c_{66}^{0,1}(\hat{u}_1^{1,1} - \theta_{,1})] \end{aligned} \tag{8.6}$$

where $\hat{u}_1^{1,1} = u_1^{1,1} + S_{23}^{0,0}$. Then, substituting the second of (8.5) and (8.6) into the first of (8.3) and solving for $\hat{u}_1^{1,1}$, we find

$$\hat{u}_1^{1,1} = [(I^{0,2}c_{66}^{0,1} - I^{2,0}c_{55}^{1,0})\theta_{,1} - I^{0,2}c_{16}^{0,1}u_{1,1}^{0,1} - I^{2,0}c_{15}^{1,0}u_{1,1}^{1,0}]/(I^{0,2}c_{66}^{0,1} + I^{2,0}c_{55}^{1,0}). \tag{8.7}$$

With this value of $\hat{u}_1^{1,1}$, we have, from (8.5) and (8.6),

$$T_1^{1,0} = \frac{I^{2,0}}{I^{0,2}c_{66}^{0,1} + I^{2,0}c_{55}^{1,0}}\left[2I^{0,2}c_{66}^{0,1}c_{15}^{1,0}\theta_{,1} + \left(\frac{I^{2,0}}{s_{11}s_{55} - s_{15}^2} + I^{0,2}c_{11}^{1,0}c_{66}^{0,1}\right)u_{1,1}^{1,0} - I^{0,2}c_{15}^{1,0}c_{16}^{0,1}u_{1,1}^{0,1}\right], \tag{8.8}$$

$$T_1^{0,1} = \frac{I^{0,2}}{I^{0,2}c_{66}^{0,1} + I^{2,0}c_{55}^{1,0}}\left[-2I^{2,0}c_{55}^{1,0}c_{16}^{0,1}\theta_{,1} + \left(\frac{I^{0,2}}{s_{11}s_{66} - s_{16}^2} + I^{2,0}c_{11}^{0,1}c_{55}^{1,0}\right)u_{1,1}^{0,1} - I^{2,0}c_{15}^{1,0}c_{16}^{0,1}u_{1,1}^{1,0}\right], \tag{8.9}$$

$$T_5^{1,0} - T_6^{0,1} = \frac{2I^{0,2}I^{2,0}(2c_{55}^{1,0}c_{66}^{0,1}\theta_{,1} + c_{15}^{1,0}c_{66}^{0,1}u_{1,1}^{1,0} - c_{16}^{0,1}c_{55}^{1,0}u_{1,1}^{0,1})}{I^{0,2}c_{66}^{0,1} + I^{2,0}c_{55}^{1,0}}, \tag{8.10}$$

where we have used, from (3.20) and (3.21),

$$\begin{aligned} c_{11}^{1,0}c_{55}^{1,0} - (c_{15}^{1,0})^2 &= (s_{11}s_{55} - s_{15}^2)^{-1},\\ c_{11}^{0,1}c_{66}^{0,1} - (c_{16}^{0,1})^2 &= (s_{11}s_{66} - s_{16}^2)^{-1}. \end{aligned}$$

Now, (8.10) is the torque M_1; but, to get the torque-twist relation in a form comparable to (8.2), we must first solve (8.8) and (8.9) for $u_{1,1}^{0,1}$ and $u_{1,1}^{1,0}$ in terms of $\theta_{,1}, T_1^{0,1}$ $(=M_2)$ and $T_1^{1,0}$ $(=-M_3)$:

$$u_{1,1}^{0,1} = \left[\frac{2I^{2,0}c_{16}^{0,1}\theta_{,1}}{s_{11}s_{55} - s_{15}^2} + \left(\frac{I^{2,0}}{I^{0,2}(s_{11}s_{55} - s_{15}^2)} + c_{11}^{1,0}c_{66}^{0,1}\right)M_2 - c_{15}^{1,0}c_{16}^{0,1}M_3\right]/\Delta,$$

$$u_{1,1}^{1,0} = \left[-\frac{2I^{0,2}c_{15}^{1,0}\theta_{,1}}{s_{11}s_{66}-s_{16}^2} - \left(\frac{I^{0,2}}{I^{2,0}(s_{11}s_{66}-s_{16}^2)} + c_{11}^{0,1}c_{55}^{1,0} \right) M_3 + c_{15}^{1,0}c_{16}^{0,1}M_2 \right] / \Delta,$$

$$\Delta = \frac{I^{0,2}c_{11}^{1,0}}{s_{11}s_{66}-s_{16}^2} + \frac{I^{2,0}c_{11}^{0,1}}{s_{11}s_{55}-s_{15}^2}.$$

Upon substituting these values in (8.10) and solving for $\theta_{,1}$, we find exactly the Voigt result (8.2).

The equations of motion are obtained by substituting (8.4), (8.8)–(8.10) in (4.1), (4.2) and (4.5), with the result

$$a_{11}u_1^{0,0} + a_{12}u_2^{0,0} + a_{13}u_3^{0,0} + a_{14}u_1^{1,0} + a_{15}u_1^{0,1} + a_{16}\theta + F_1^{0,0} = \rho I^{0,0}\ddot{u}_1^{0,0}, \tag{8.11}$$

$$a_{21}u_1^{0,0} + a_{22}u_2^{0,0} + a_{23}u_3^{0,0} + a_{24}u_1^{1,0} + a_{25}u_1^{0,1} + a_{26}\theta + F_2^{0,0} = \rho I^{0,0}\ddot{u}_2^{0,0}, \tag{8.12}$$

$$a_{31}u_1^{0,0} + a_{32}u_2^{0,0} + a_{33}u_3^{0,0} + a_{34}u_1^{1,0} + a_{35}u_1^{0,1} + a_{36}\theta + F_3^{0,0} = \rho I^{0,0}\ddot{u}_3^{0,0}, \tag{8.13}$$

$$a_{41}u_1^{0,0} + a_{42}u_2^{0,0} + a_{43}u_3^{0,0} + a_{44}u_1^{1,0} + a_{45}u_1^{0,1} + a_{46}\theta - F_1^{1,0} = 0, \tag{8.14}$$

$$a_{51}u_1^{0,0} + a_{52}u_2^{0,0} + a_{53}u_3^{0,0} + a_{54}u_1^{1,0} + a_{55}u_1^{0,1} + a_{56}\theta - F_1^{0,1} = 0, \tag{8.15}$$

$$a_{61}u_1^{0,0} + a_{62}u_2^{0,0} + a_{63}u_3^{0,0} + a_{64}u_1^{1,0} + a_{65}u_1^{0,1} + a_{66}\theta - F_\theta = -\rho(I^{2,0}+I^{0,2})\ddot{\theta}, \tag{8.16}$$

where $a_{pq} = a_{qp}$ and

$$\begin{aligned}
&a_{11} = I^{0,0}c_{11}^{0,0}\partial^2 \\
&a_{12} = I^{0,0}c_{16}^{0,0}\partial^2 \quad a_{22} = I^{0,0}c_{66}^{0,0}\partial^2 \\
&a_{13} = I^{0,0}c_{15}^{0,0}\partial^2 \quad a_{23} = I^{0,0}c_{56}^{0,0}\partial^2 \quad a_{33} = I^{0,0}c_{55}^{0,0}\partial^2 \\
&a_{14} = I^{0,0}c_{16}^{0,0}\partial \quad a_{24} = I^{0,0}c_{66}^{0,0}\partial \quad a_{34} = I^{0,0}c_{56}^{0,0}\partial \\
&a_{15} = I^{0,0}c_{15}^{0,0}\partial \quad a_{25} = I^{0,0}c_{56}^{0,0}\partial \quad a_{35} = I^{0,0}c_{55}^{0,0}\partial \\
&a_{16} = 0 \quad a_{26} = 0 \quad a_{36} = 0 \\
&a_{44} = -\alpha I^{2,0}\{I^{2,0}[c_{11}^{1,0}c_{55}^{1,0} - (c_{15}^{1,0})^2] + I^{0,2}c_{11}^{1,0}c_{66}^{0,1}\}\partial^2 + I^{0,0}c_{66}^{0,0} \\
&a_{45} = \alpha I^{2,0}I^{0,2}c_{15}^{1,0}c_{16}^{0,1}\partial^2 + I^{0,0}c_{56}^{0,0} \\
&a_{46} = -2\alpha I^{2,0}I^{0,2}c_{15}^{1,0}c_{66}^{0,1}\partial^2 \\
&a_{55} = -\alpha I^{0,2}\{I^{0,2}[c_{11}^{0,1}c_{66}^{0,1} - (c_{16}^{0,1})^2] + I^{2,0}c_{11}^{0,1}c_{55}^{1,0}\}\partial^2 + I^{0,0}c_{55}^{0,0} \\
&a_{56} = 2\alpha I^{2,0}I^{0,2}c_{16}^{0,1}c_{55}^{1,0}\partial^2 \\
&a_{66} = -4\alpha I^{2,0}I^{0,2}c_{55}^{1,0}c_{66}^{0,1}\partial^2 \\
&\alpha = (I^{0,2}c_{66}^{0,1} + I^{2,0}c_{55}^{1,0})^{-1}
\end{aligned}$$

Recalling the definitions of $c_{\alpha\beta}^{0,0}$, $c_{ab}^{1,0}$ and $c_{cd}^{0,1}$ in (3.9), (3.20) and (3.21), it may be seen that all the coupling between modes is solely through the compliances s_{15}, s_{16} and s_{56}. The connections are depicted in Table 2. In general, extension, both flexures and torsion are coupled; although, since $a_{16} = 0$, the coupling of torsion with extension is not direct, but through flexure. Torsion (θ) is coupled with the rotational component $u_1^{1,0}$ of flexure in the $x_1 - x_2$ plane through a_{46}, i.e. through s_{15}, and with the rotational component $u_1^{0,1}$ of flexure in the $x_1 - x_3$ plane through a_{56}, i.e. through s_{16}. Extension is coupled with both components, $u_2^{0,0}$ and $u_1^{1,0}$, of flexure in the $x_1 - x_2$ plane through a_{12} and a_{14}, respectively, i.e. through s_{16}; and with both components, $u_3^{0,0}$ and $u_1^{0,1}$, of flexure in the $x_1 - x_3$ plane through a_{13} and a_{15}, respectively, i.e. through s_{15}. The two flexures are coupled through s_{56}.

If s_{56} is not zero, but both s_{15} and s_{16} are zero—as, for example, in the case of monoclinic symmetry with x_1 the digonal axis, for which[24]

$$s_{15} = s_{16} = s_{25} = s_{26} = s_{35} = s_{36} = s_{45} = s_{46} = 0 \tag{8.17}$$

Table 2. Coupling coefficients: anisotropic bar, elliptic section

		Extension	Flexure $x_1 - x_2$		Flexure $x_1 - x_3$		Torsion
		$u_1^{0,0}$	$u_2^{0,0}$	$u_1^{1,0}$	$u_3^{0,0}$	$u_1^{0,1}$	θ
Extension $u_1^{0,0}$		(8.11)	a_{12} s_{16}	a_{14} s_{16}	a_{13} s_{15}	a_{15} s_{15}	
Flexure $x_1 - x_2$	$u_2^{0,0}$	a_{12} s_{16}	(8.12)	a_{24} $c_{66}^{0,0}$	a_{23} s_{56}	a_{25} s_{56}	
	$u_1^{1,0}$	a_{14} s_{16}	a_{24} $c_{66}^{0,0}$	(8.14)	a_{34} s_{56}	a_{45} s_{56}	a_{46} s_{15}
Flexure $x_1 - x_3$	$u_3^{0,0}$	a_{13} s_{15}	a_{23} s_{56}	a_{34} s_{56}	(8.13)	a_{35} $c_{55}^{0,0}$	
	$u_1^{0,1}$	a_{15} s_{15}	a_{25} s_{56}	a_{45} s_{56}	a_{35} $c_{55}^{0,0}$	(8.15)	a_{56} s_{16}
Torsion θ				a_{46} s_{15}		a_{56} s_{16}	(8.16)

the two flexures are coupled, but torsion and extension are independent. If s_{56} is not zero and either s_{15} or s_{16} is not zero, all four modes of motion are coupled. For example, suppose that s_{56} and s_{15} are not zero, but s_{16} is zero, as is the case of classes 3 and $\bar{3}$ of the trigonal system with x_3 the trigonal axis, in which case[24]

$$\begin{gathered} s_{16} = s_{26} = s_{34} = s_{35} = s_{36} = s_{45} = 0, \\ s_{11} = s_{22}, \quad s_{13} = s_{23}, \quad s_{44} = s_{55}, \quad s_{66} = 2(s_{11} - s_{12}), \\ s_{25} = -s_{15} = \frac{1}{2} s_{46}, \quad s_{14} = -s_{24} = \frac{1}{2} s_{56}. \end{gathered} \tag{8.18}$$

Then, with x_1 along the axis of the bar, extension is coupled, through a_{13} and a_{15}, with $x_1 - x_3$ flexure which is coupled, through a_{23}, a_{25}, a_{43} and a_{45}, with $x_1 - x_2$ flexure which is coupled, through a_{46}, with torsion.

If s_{56} is zero, crystal symmetry conditions require that either or both of s_{15} and s_{16} must be zero and thus a_{23}, a_{25}, a_{43} and a_{45}, which depend on s_{56} and the product of s_{15} and s_{16}, must be zero. Consequently, if $s_{56} = 0$, the coupling between the two flexures is absent. If s_{56} and only one of s_{15} and s_{16} are zero, there is coupling between extension and one of the flexures and between torsion and the other flexure, but no coupling between the two pairs. For example, if it is s_{15} that is zero, as in the case of classes, 4, $\bar{4}$ and $4/m$ of the tetragonal system, for which[24]

$$\begin{gathered} s_{14} = s_{15} = s_{24} = s_{25} = s_{34} = s_{35} = s_{36} = s_{45} = s_{46} = s_{56} = 0 \\ s_{11} = s_{22}, \quad s_{23} = s_{13}, \quad s_{26} = -s_{16}, \quad s_{44} = s_{55} \end{gathered} \tag{8.19}$$

if x_3 is the tetragonal axis, then extension is coupled with flexure in the $x_1 - x_2$ plane and torsion is coupled with flexure in the $x_1 - x_3$ plane.

In all of the foregoing, the axis of the bar is that of x_1. If the axis of the bar is that of x_2 or x_3, subscripts 1, 2, 3 are permuted cyclically and, independently, subscripts 4, 5, 6 are permuted cyclically.

Reduction of the ellipse to a circle does not affect the coupling.

9. QUARTZ BAR, ELLIPTIC SECTION

As an illustration of cyclical permutation, when the axis of the bar is not along x_1, and as an instance of coupling between torsion and one flexure and between extension and the other flexure, but no coupling between the two pairs, consider the case of an alpha-quartz bar of elliptic section with x_3 the trigonal axis, x_1 a digonal axis and x_2 the axis of centroids of the sections. The restrictions on the compliances, for this case of crystallographic class 32 (and also $3m$ and $\bar{3}m$ in the absence of piezoelectric properties) of the trigonal system, are[24]

$$
\begin{gathered}
s_{15}=s_{16}=s_{25}=s_{26}=s_{34}=s_{35}=s_{36}=s_{45}=s_{46}=0,\\
s_{11}=s_{22},\quad s_{23}=s_{31},\quad s_{24}=-s_{14},\\
s_{44}=s_{55},\quad s_{56}=2s_{14},\quad s_{66}=2(s_{11}-s_{12}).
\end{gathered}\tag{9.1}
$$

As x_2 is along the axis of the bar, s_{26}, s_{24} and s_{64} replace s_{15}, s_{16} and s_{56}, respectively, in Table 2 and, from (9.1), s_{26} and s_{64} are zero but s_{24} is not. Furthermore, in Table 2, $u_1^{0,0}$, $u_2^{0,0}$, $u_3^{0,0}$, $u_1^{1,0}$, $u_1^{0,1}$ become $u_2^{0,0}$, $u_3^{0,0}$, $u_1^{0,0}$, $u_2^{1,0}$, $u_2^{0,1}$, respectively. Thus, the coupling is between extension ($u_2^{0,0}$) and flexure ($u_3^{0,0}$, $u_2^{1,0}$) in the x_2-x_3 plane and between torsion (θ) and flexure ($u_1^{0,0}$, $u_2^{0,1}$) in the x_2-x_1 plane.

The equations of motion of the "extensional" and "torsional" groups become, respectively,

$$
\begin{aligned}
&c_{22}^{0,0}\partial^2u_2^{0,0}+c_{24}^{0,0}\partial^2u_3^{0,0}+c_{24}^{0,0}\partial u_2^{1,0}+F_2^{0,0}/I^{0,0}=\rho\ddot{u}_2^{0,0},\\
&c_{24}^{0,0}\partial^2u_2^{0,0}+c_{44}^{0,0}\partial^2u_3^{0,0}+c_{44}^{0,0}\partial u_2^{1,0}+F_3^{0,0}/I^{0,0}=\rho\ddot{u}_3^{0,0},\\
&c_{24}^{0,0}\partial u_2^{0,0}+c_{44}^{0,0}\partial u_3^{0,0}+[c_{44}^{0,0}-c_{22}^{1,0}(I^{2,0}/I^{0,0})\partial^2]u_2^{1,0}+F_1^{1,0}/I^{0,0}=0,
\end{aligned}\tag{9.2}
$$

and

$$
\begin{aligned}
&c_{66}^{0,0}\partial^2u_1^{0,0}+c_{66}^{0,0}\partial u_2^{0,1}+F_1^{0,0}=\rho\ddot{u}_1^{0,0}\\
&c_{66}^{0,0}\partial u_1^{0,0}-\alpha(I^{0,2}/I^{0,0})\{I^{0,2}[c_{22}^{0,1}c_{44}^{0,1}-(c_{24}^{0,1})^2]+I^{2,0}c_{22}^{0,1}c_{66}^{1,0}\}\partial^2u_2^{0,1},\\
&\qquad+c_{66}^{0,0}u_2^{0,1}+2\alpha(I^{2,0}I^{0,2}/I^{0,0})c_{24}^{0,1}c_{66}^{1,0}\partial^2\theta-F_2^{0,1}/I^{0,0}=0,\\
&2\alpha I^{2,0}I^{0,2}c_{24}^{0,1}c_{66}^{1,0}\partial^2u_2^{0,1}-4\alpha I^{2,0}I^{0,2}c_{66}^{1,0}c_{44}^{0,1}\partial^2\theta-F_\theta=-\rho(I^{2,0}+I^{0,2})\ddot{\theta},
\end{aligned}\tag{9.3}
$$

where $\alpha=(I^{0,2}c_{44}^{0,1}+I^{2,0}c_{66}^{1,0})^{-1}$.

Considering the extensional group (9.2), we set $F_1^{0,0}$, $F_2^{0,0}$, $F_1^{1,0}$ equal to zero and

$$
(u_2^{0,0},u_3^{0,0},u_2^{1,0})=(A,B,C)e^{i(\eta x_2-\omega t)}\tag{9.4}
$$

and find

$$
\begin{aligned}
&(\rho\omega^2-\eta^2c_{22}^{0,0})A-\eta^2c_{24}^{0,0}B+i\eta c_{24}^{0,0}C=0,\\
&-\eta^2c_{24}^{0,0}A+(\rho\omega^2-\eta^2c_{44}^{0,0})B+i\eta c_{44}^{0,0}C=0,\\
&i\eta c_{24}^{0,0}A+i\eta c_{44}^{0,0}B+(c_{44}^{0,0}+c_{22}^{1,0}\eta^2I^{2,0}/I^{0,0})C=0,
\end{aligned}\tag{9.5}
$$

The dispersion relation is obtained by setting the determinant of the coefficients of A, B, C equal to zero, resulting in a biquadratic equation on the frequency:

$$
[1+(s_{22}s_{44}-s_{24}^2)\bar{\eta}^2/s_{22}^2]\Omega^4-[1+\bar{\eta}^2(1+s_{44}/s_{22})]\bar{\eta}^2\Omega^2+\bar{\eta}^6=0\tag{9.6}
$$

or, alternatively, a bicubic on the wave number:

$$
\bar{\eta}^6+(1+s_{44}/s_{22})\Omega^2\bar{\eta}^4-[1-(s_{22}s_{44}-s_{24}^2)\Omega^2/s_{22}^2]\Omega^2\bar{\eta}^2+\Omega^4=0,\tag{9.7}
$$

where

$$
\Omega^2=\rho\omega^2h^2s_{22},\quad \bar{\eta}=\eta h,\quad h^2=I^{2,0}/I^{0,0}=b^2/4
$$

and we note that, in the present case of trigonal symmetry,

$$
\begin{gathered}
(c_{22}^{0,0},c_{44}^{0,0},c_{24}^{0,0})=(s_{44},s_{22},s_{24})/(s_{22}s_{44}-s_{24}^2),\\
c_{22}^{1,0}=[c_{22}^{0,0}c_{44}^{0,0}-(c_{24}^{0,0})^2]/c_{44}^{0,0}=1/s_{22}.
\end{gathered}
$$

The coupling between extension and flexure is through s_{24}. If s_{24} is set equal to zero, (9.6) and (9.7) reduce to

$$(\Omega^2-\bar{\eta}^2)\,[\Omega^2-\bar{\eta}^4/(1+s_{44}\bar{\eta}^2/s_{22})]=0. \tag{9.8}$$

The first factor in (9.8) gives Bernoulli's result for dispersionless longitudinal waves; the second factor gives the Bernoulli–Euler dispersion relation for low frequency flexural waves with Timoshenko's correction for shear deformation: $s_{44}\bar{\eta}^2/s_{22}$.

In the case of a bar of finite length, the boundary conditions most convenient for reproduction in the laboratory (but, unfortunately, not the simplest mathematically) are those for free ends:

$$T_2^{0,0}=T_4^{0,0}=T_2^{1,0}=0 \quad \text{on} \quad x_2=\pm l \tag{9.9}$$

or, again on $x_2=\pm l$, from (8.4) and (8.5) with one cyclical permutation of subscripts:

$$\begin{aligned} c_{22}^{0,0}u_{2,2}^{0,0}+c_{24}^{0,0}u_{3,2}^{0,0}+c_{24}^{0,0}u_2^{1,0}&=0,\\ c_{24}^{0,0}u_{2,2}^{0,0}+c_{44}^{0,0}u_{3,2}^{0,0}+c_{44}^{0,0}u_2^{1,0}&=0,\\ u_{2,2}^{1,0}&=0. \end{aligned} \tag{9.10}$$

The three conditions (9.10) are such that all three roots $\bar{\eta}^2$, of (9.7), are required for each Ω. Two of these roots are positive and one negative, so that two of the η are real and one is imaginary. For every Ω and each η_i, $i=1,2,3$, the simultaneous equations (9.5) define a set of amplitude ratios A:B:C. For each η_i, let

$$\alpha_{2i}=B/A, \quad \alpha_{3i}=C/A. \tag{9.11}$$

Then (9.4) may be written as

$$\begin{aligned} u_2^{0,0}&=h(A_1\sin\eta_1x_2+A_2\sin\eta_2x_2+A_3\sinh\eta_3x_2),\\ u_3^{0,0}&=h(A_1\alpha_{21}\sin\eta_1x_2+A_2\alpha_{22}\sin\eta_2x_2+A_3\alpha_{23}\sinh\eta_3x_2),\\ u_2^{1,0}&=A_1\alpha_{31}\cos\eta_1x_2+A_2\alpha_{32}\cos\eta_2x_2+A_3\alpha_{33}\cosh\eta_3x_2, \end{aligned} \tag{9.12}$$

with

$$\begin{aligned} \alpha_{2i}&=\frac{s_{22}}{s_{24}}\left(1\mp\frac{\bar{\eta}_i^2}{\Omega^2}\right),\\ \alpha_{3i}&=\pm\frac{(s_{22}s_{44}-s_{24}^2)\Omega^2}{s_{22}s_{24}\bar{\eta}_i}-\frac{(s_{22}+s_{44})\bar{\eta}_i}{s_{24}}\pm\frac{s_{22}\bar{\eta}_i^3}{s_{24}\Omega^2}, \end{aligned} \tag{9.13}$$

where the upper signs are for $i=1,2$ and the lower signs are for $i=3$.

In (9.12) the extensional motion is symmetric and the flexural motion is antisymmetric with respect to the center of the bar. The converse set would be obtained by interchanging sine and cosine (trigonometric and hyperbolic).

Upon substituting (9.12) in the boundary conditions (9.10), we find

$$\begin{aligned} A_1\beta_{11}\cos\eta_1l+A_2\beta_{12}\cos\eta_2l+A_3\beta_{13}\cosh\eta_3l&=0,\\ A_1\beta_{21}\cos\eta_1l+A_2\beta_{22}\cos\eta_2l+A_3\beta_{23}\cosh\eta_3l&=0,\\ A_1\beta_{31}\sin\eta_1l+A_2\beta_{32}\sin\eta_2l+A_3\beta_{33}\sinh\eta_3l&=0 \end{aligned} \tag{9.14}$$

where

$$\begin{aligned} \beta_{1i}&=[s_{44}\bar{\eta}_i+s_{24}(\bar{\eta}_i\alpha_{2i}+\alpha_{3i})]/(s_{22}s_{44}-s_{24}^2),\\ \beta_{2i}&=[s_{24}\bar{\eta}_i+s_{22}(\bar{\eta}_i\alpha_{2i}+\alpha_{3i})]/(s_{22}s_{44}-s_{24}^2),\\ \beta_{3i}&=\bar{\eta}_i\alpha_{3i}. \end{aligned} \tag{9.15}$$

Finally, the frequency equation is obtained by setting the determinant of the coefficients of the A_i, in (9.14), equal to zero:

$$B_1 \tan \eta_1 l + B_2 \tan \eta_2 l + B_3 \tanh \eta_3 l = 0, \tag{9.16}$$

where

$$\begin{aligned} B_1 &= \beta_{31}(\beta_{12}\beta_{23} - \beta_{22}\beta_{13}), \\ B_2 &= \beta_{32}(\beta_{13}\beta_{21} - \beta_{23}\beta_{11}), \\ B_3 &= \beta_{33}(\beta_{11}\beta_{22} - \beta_{21}\beta_{12}). \end{aligned} \tag{9.17}$$

The solution for the torsional group (9.3) is similar as it involves only coupling between a dispersionless mode (in this case torsion instead of extension) and a flexural mode (in this case in the $x_2 - x_1$ plane instead of the $x_2 - x_3$ plane) including the Timoshenko shear correction. Since, in (9.3),

$$c_{24}^{0,1} = -s_{24}/(s_{22}s_{44} - s_{24}^2),$$

the coupling coefficient again depends on s_{24}.

10. ANISOTROPIC BARS: RESTRICTIONS

There are certain situations in which the contour stresses (T_{22}, T_{33}, T_{23}, in a bar with axis along x_1) do not vanish as the frequency approaches zero; in which cases the equations deduced in Sections 3 and 4 are not valid. The survival of contour stresses in torsional equilibrium may be seen by inspecting Voigt's compatibility equations[18, p. 641] which must be satisfied by a St. Venant torsion function (φ) and the Airy plane strain function (χ) for St. Venant-type torsion of anisotropic bars.

If x_1 is the axis of centroids of the cross-sections of the bar and if the strains are independent of x_1, the two stress functions are defined by

$$\begin{aligned} T_{31} &= -\frac{\partial \varphi}{\partial x_2}, \quad T_{12} = \frac{\partial \varphi}{\partial x_3} \\ T_{22} &= \frac{\partial^2 \chi}{\partial x_3^2}, \quad T_{33} = \frac{\partial^2 \chi}{\partial x_2^2}, \quad T_{23} = -\frac{\partial^2 x}{\partial x_2 \partial x_3} \end{aligned} \tag{10.1}$$

in consequence of which the equilibrium equations

$$T_{ij,i} = 0 \tag{10.2}$$

are satisfied. The strains, related to the stresses according to

$$S_{ij} = s_{ijkl} T_{kl}, \tag{10.3}$$

must satisfy the compatibility equations

$$\epsilon_{ikm}\epsilon_{jln} S_{ij,kl} = 0 \tag{10.4}$$

where ϵ_{ijk} is the unit alternating tensor. Upon substituting (10.1) in (10.3) and the result in (10.4), we find Voigt's compatibility equations on φ and χ [18, p. 643]:

$$\begin{aligned} D_{11}\varphi + D_{12}\chi &= -2\theta_{,1} - (s_{15}\kappa_2 - s_{16}\kappa_3)/s_{11}, \\ D_{12}\varphi + D_{22}\chi &= 0, \end{aligned} \tag{10.5}$$

where κ_2 and κ_3 are the curvatures in the $x_1 - x_2$ and $x_2 - x_3$ planes and

$$D_{11} = \bar{s}_{55}\frac{\partial^2}{\partial x_2^2} - 2\bar{s}_{56}\frac{\partial^2}{\partial x_2 \partial x_3} + \bar{s}_{66}\frac{\partial^2}{\partial x_3^2},$$

$$D_{22} = \bar{s}_{33}\frac{\partial^4}{\partial x_2^4} - 2\bar{s}_{34}\frac{\partial^4}{\partial x_2^3\partial x_3} + (2\bar{s}_{23} + \bar{s}_{44})\frac{\partial^4}{\partial x_2^2\partial x_3^2} - 2\bar{s}_{24}\frac{\partial^4}{\partial x_2\partial x_3^3} + \bar{s}_{22}\frac{\partial^3}{\partial x_3^4}, \tag{10.6}$$

$$D_{12} = -\bar{s}_{35}\frac{\partial^3}{\partial x_2^3} + (\bar{s}_{45} + \bar{s}_{36})\frac{\partial^3}{\partial x_2^2\partial x_3} - (\bar{s}_{25} + \bar{s}_{46})\frac{\partial^3}{\partial x_2\partial x_3^2} + \bar{s}_{26}\frac{\partial^3}{\partial x_3^3},$$

in which

$$\bar{s}_{rs} = s_{rs} - s_{r1}s_{1s}/s_{11}; \quad r,s = 2,3,4,5,6. \tag{10.7}$$

It may be seen that, if $D_{12}\varphi = 0$, χ may be taken to be zero so that there need be no contour stresses accompanying torsion and flexure. This is the case if φ is quadratic in x_2 and x_3, as it is for the elliptic section, or if

$$\bar{s}_{25} + \bar{s}_{46} = \bar{s}_{45} + \bar{s}_{36} = \bar{s}_{35} = \bar{s}_{26} = 0. \tag{10.8}$$

It is unlikely that (10.8) would be satisfied unless at least the line of centroids of the cross-sections is an axis of two-fold symmetry, in which case

$$s_{15} = s_{16} = s_{25} = s_{26} = s_{35} = s_{36} = s_{45} = s_{46} = 0. \tag{10.9}$$

Thus, for the truncated equations to hold for anisotropic bars, the normal section must be either elliptic or a plane of elastic symmetry.

11. ANISOTROPIC BAR, RECTANGULAR SECTION

The rectangular section is bounded by $x_2 = \pm b$, $x_3 = \pm c$, as in Article 7, but now the material is anisotropic with x_1 a digonal axis of symmetry so that (10.9) hold and (3.8), (3.20), (3.21) and (3.29) reduce to

$$\begin{gathered} c_{15}^{0,0} = c_{16}^{0,0} = c_{15}^{1,0} = c_{16}^{0,1} = 0, \\ c_{11}^{0,0} = c_{11}^{1,0} = c_{11}^{0,1} = 1/s_{11}, \quad c_{55}^{1,0} = 1/s_{55}, \quad c_{66}^{0,1} = 1/s_{66}, \\ (c_{55}^{0,0}, c_{66}^{0,0}, c_{56}^{0,0}) = (\bar{c}_{55}, \bar{c}_{66}, \bar{c}_{56}) = (s_{66}, s_{55}, -s_{56})/\Delta_{56}, \\ \Delta_{56} = s_{55}s_{66} - s_{56}^2. \end{gathered} \tag{11.1}$$

Again, from the $u_1^{m,n}$, $m+n>1$, we retain only $u_1^{1,1}$, $u_1^{1,3}$, $u_1^{3,1}$ and $u_1^{3,3}$ along with the corresponding equilibrium equations (7.2). Then, employing (11.1), we find, from (3.7), (3.11), (3.12) and (3.13), the zero order stresses

$$\begin{aligned} T_1^{0,0} &= I^{0,0}u_{1,1}^{0,0}/s_{11}, \\ T_5^{0,0} &= I^{0,0}[s_{66}(u_{3,1}^{0,0} + u_1^{0,1}) - s_{56}(u_{2,1}^{0,0} + u_1^{1,0})]/\Delta_{56}, \\ T_6^{0,0} &= I^{0,0}[s_{55}(u_{2,1}^{0,0} + u_1^{1,0}) - s_{56}(u_{3,1}^{0,0} + u_1^{0,1})]/\Delta_{56}, \end{aligned}$$

and, from (3.19), (3.15) and (3.16), the first order stresses

$$T_1^{0,1} = I^{0,2}u_{1,1}^{0,1}/s_{11}, \quad T_1^{1,0} = I^{2,0}u_{1,1}^{1,0}/s_{11}, \tag{11.3}$$

and

$$\begin{aligned} s_{66}T_6^{0,1} &= -I^{0,2}\theta_{,1} + I^{0,2}\hat{u}_1^{1,1} + I^{0,4}u_1^{1,3} + 3I^{2,2}u_1^{3,1} + 3I^{2,4}u_1^{3,3}, \\ s_{55}T_5^{1,0} &= I^{2,0}\theta_{,1} + I^{2,0}\hat{u}_1^{1,1} + 3I^{2,2}u_1^{1,3} + I^{4,0}u_1^{3,1} + 3I^{4,2}u_1^{3,3}, \end{aligned} \tag{11.4}$$

in which

$$\hat{u}_1^{1,1} = u_1^{1,1} + s_{14}u_{1,1}^{0,0}/s_{11} \tag{11.5}$$

and we have employed (4.4), (3.14) and (3.7).

For the higher order stresses, we find, from (3.28),

$$\begin{aligned}
s_{66}T_6^{0,3} &= I^{0,4}\theta_{,1} + I^{0,4}\hat{u}_1^{1,1} + (s_{55}s_{56}/\Delta_{56})(I^{0,6}u_1^{1,3} + 3I^{2,4}u_1^{3,1} + 3I^{2,6}u_1^{3,3}) - (s_{56}^2 I^{0,4}/\Delta_{56}I^{0,2}) \\
&\quad \times (I^{0,4}u_1^{1,3} + 3I^{2,2}u_1^{3,1} + 3I^{2,4}u_1^{3,3}), \\
s_{55}T_5^{1,2} &= I^{2,2}\theta_{,1} + I^{2,2}\hat{u}_1^{1,1} + (s_{55}s_{56}/\Delta_{56})(3I^{2,4}u_1^{1,3} + I^{4,2}u_1^{3,1} + 3I^{4,4}u_1^{3,3}) - (s_{56}^2 I^{2,2}/\Delta_{56}I^{2,0}) \\
&\quad \times (3I^{2,2}u_1^{1,3} + I^{4,0}u_1^{3,1} + 3I^{4,2}u_1^{3,3}), \\
s_{66}T_6^{2,1} &= -I^{2,2}\theta_{,1} + I^{2,2}\hat{u}_1^{1,1} + (s_{55}s_{66}/\Delta_{56})(I^{2,4}u_1^{1,3} + 3I^{4,2}u_1^{3,1} + 3I^{4,4}u_1^{3,3}) - (s_{56}^2 I^{2,2}/\Delta_{56}I^{0,2}) \\
&\quad \times (I^{0,4}u_1^{1,3} + 3I^{2,2}u_1^{3,1} + 3I^{2,4}u_1^{3,3}), \\
s_{55}T_5^{3,0} &= I^{4,0}\theta_{,1} + I^{4,0}\hat{u}_1^{1,1} + (s_{55}s_{66}/\Delta_{56})(3I^{4,2}u_1^{1,3} + I^{6,0}u_1^{3,1} + 3I^{6,2}u_1^{3,3}) - (s_{56}^2 I^{4,0}/\Delta_{56}I^{2,0}) \\
&\quad \times (3I^{2,2}u_1^{1,3} + I^{4,0}u_1^{3,1} + 3I^{4,2}u_1^{3,3}), \\
s_{66}T_6^{2,3} &= -I^{2,4}\theta_{,1} + I^{2,4}\hat{u}_1^{1,1} + (s_{55}s_{66}/\Delta_{56})(I^{2,6}u_1^{1,3} + 3I^{4,4}u_1^{3,1} + 3I^{4,6}u_1^{3,3}) - (s_{56}^2 I^{2,4}/\Delta_{56}I^{0,2}) \\
&\quad \times (I^{0,4}u_1^{1,3} + 3I^{2,2}u_1^{3,1} + 3I^{2,4}u_1^{3,3}), \\
s_{55}T_5^{3,2} &= I^{4,2}\theta_{,1} + I^{4,2}\hat{u}_1^{1,1} + (s_{55}s_{56}/\Delta_{56})(3I^{4,4}u_1^{1,3} + I^{6,2}u_1^{3,1} + 3I^{6,4}u_1^{3,3}) - (s_{56}^2 I^{4,2}/\Delta_{56}I^{2,0}) \\
&\quad \times (3I^{2,2}u_1^{1,3} + I^{4,0}u_1^{3,1} + 3I^{4,2}u_1^{3,3}).
\end{aligned} \tag{11.6}$$

The equations of extensional and flexural motions are obtained by substituting (11.2) and (11.3) in (4.1) and (4.2) with the results:

Extensional:

$$(I^{0,0}/s_{11})u_{1,11}^{0,0} + F_1^{0,0} = \rho I^{0,0}\ddot{u}_1^{0,0} \tag{11.7}$$

Flexural ($x_1 - x_2$ *plane*):

$$\begin{aligned}
&(I^{0,0}/\Delta_{56})\,[s_{55}(u_{2,11}^{0,0} + u_{1,1}^{1,0}) - s_{56}(u_{3,11}^{0,0} + u_{1,1}^{0,1})] + F_2^{0,0} = \rho I^{0,0}\ddot{u}_2^{0,0}, \\
&(I^{2,0}/s_{11})u_{1,11}^{1,0} - (I^{0,0}/\Delta_{56})\,[s_{55}(u_{2,1}^{0,0} + u_1^{1,0}) - s_{56}(u_{3,1}^{0,0} + u_1^{0,1})] + F_1^{1,0} = 0
\end{aligned} \tag{11.8}$$

Flexural ($x_1 - x_3$ *plane*):

$$\begin{aligned}
&(I^{0,0}/\Delta_{56})\,[s_{66}(u_{3,11}^{0,0} + u_{1,1}^{0,1}) - s_{56}(u_{2,11}^{0,0} + u_{1,1}^{1,0})] + F_3^{0,0} = \rho I^{0,0}\ddot{u}_3^{0,0} \\
&(I^{0,2}/s_{11})u_{1,11}^{0,1} - (I^{0,0}/\Delta_{56})\,[s_{66}(u_{3,1}^{0,0} + u_1^{0,1}) - s_{56}(u_{2,1}^{0,0} + u_1^{1,0})] + F_1^{0,1} = 0.
\end{aligned} \tag{11.9}$$

Thus, the extensional mode is independent and the two flexural modes are coupled through s_{56}.

To find the appropriate form of the equation of torsional motion, we must first express the torque in terms of the twist. This may be done, as in Section 7, by substituting the stresses (11.4) and (11.6), corresponding to (7.4), in (7.2) and solving for $\hat{u}_1^{1,1}$, $u_1^{1,3}$, $u_1^{3,1}$ and $u_1^{3,3}$ in terms of $\theta_{,1}$, corresponding to (7.5). These solutions may then be inserted in (11.4) from which the torque, $T_5^{1,0} - T_6^{0,1}$, may be expressed in terms of the twist $\theta_{,1}$. As in the case of the St. Venant theory of torsional equilibrium ([19], p. 324), the algebra is simplified in the case $s_{56} = 0$; for then the solutions for $\hat{u}_1^{1,1}$, $u_1^{1,3}$, $u_1^{3,1}$ and $u_1^{3,3}$, in terms of $\theta_{,1}$ become the same as (7.5) except that $u_1^{1,1}$ is replaced by $\hat{u}_1^{1,1}$ and k^2 is replaced by

$$\hat{k} = s_{55}c^2/s_{66}b^2. \tag{11.10}$$

As a result, the expression for the torsional rigidity, analogous to (7.7) is

$$\hat{C} = (T_5^{1,0} - T_6^{0,1})/\theta_{,1} = \hat{k}_1 s_{55}^{-1}(2b)^3(2c), \tag{11.11}$$

where $\hat{k}_1$ is given by (7.8) with k replaced by $\hat{k}$. Thus Table 1 holds for the case $s_{56} = 0$ if k and k_1 are replaced by $\hat{k}$ and $\hat{k}_1$.

Finally, the equation of torsional motion (for the case $s_{56} = 0$) is, from (4.5) and (11.11),

$$16\hat{k}_1 s_{55}^{-1} b^3 c\theta_{,11} + F_\theta = 2bc(b^2+c^2)\rho\ddot{\theta}. \tag{11.12}$$

It may be observed that the torsional mode is independent of extension (as well as flexure) despite the appearance of $u_1^{0,0}$ in (11.5), as the whole of $\hat{u}_1^{1,1}$ is eliminated from $T_5^{1,0}$ and $T_6^{0,1}$ through the equilibrium equations (7.2). This property holds whether or not $s_{56}=0$.

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